

# COCYCLE SUPERRIGIDITY AND HARMONIC MAPS WITH INFINITE DIMENSIONAL TARGETS

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ABSTRACT. We announce a generalization of Zimmer’s cocycle superrigidity theorem proven using harmonic map techniques. This allows us to generalize many results concerning higher rank lattices to all lattices in semisimple groups with property  $(T)$ . In particular, our results apply to  $SP(1, n)$  and  $F_4^{-20}$  and lattices in those groups.

The main technical step is to prove a very general result concerning existence of harmonic maps into infinite dimensional spaces, namely a class of simply connected, homogeneous, aspherical Hilbert manifolds. This builds on previous work of Corlette-Zimmer and Korevaar-Schoen. Our result is new because we consider more general targets than previous authors and make no assumption concerning the action. In particular, the target space is not assumed to have non-positive curvature and in important cases has significant positive curvature. We also do not make a “reductivity” assumption concerning absence of fixed points at infinity.

The proof of cocycle superrigidity given here, unlike previous ergodic theoretic ones, is effective. The straightening of the cocycle is explicitly a limit of a heat flow. This explicit construction should yield further applications.

## 1. INTRODUCTION

Let  $G$  be a semisimple Lie group with all simple factors of real rank at least two,  $\Gamma < G$  a lattice,  $S$  a compact manifold and  $\nu$  a volume form on  $S$ . In [Z6], Zimmer proposed a program to study volume preserving actions of  $\Gamma$  on  $S$ , with the aim of classifying homomorphisms  $\rho : \Gamma \rightarrow \text{Diff}_\nu^\infty(S)$ . A main impetus for this program was his *cocycle superrigidity theorem* which gives strong information concerning the associated action of  $\Gamma$  on any natural principal bundle over  $S$ , see [Z1, Z2, Z6, Z7, Z8] and also [Fe, FM1] for more discussion.

Zimmer’s cocycle superrigidity theorem is profoundly influenced in both statement and proof by Margulis superrigidity theorem for finite dimensional representations of  $\Gamma$  [M1]. As a consequence of Margulis’ theorem, one has an essentially complete classification of representations  $\Gamma \rightarrow GL(n, k)$  where  $k$  is a local field. Zimmer’s program can be viewed as an attempt to generalize this problem to consider *non-linear representations* instead.

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In the classical setting of Margulis' superrigidity theorems, the results have been extended by work of Corlette and Gromov-Schoen [Co2, GS] to cover the case where  $G$  is merely a semisimple Lie group with no compact factors which has property (T) of Kazhdan. The key addition here is that one is now allowed to have factors of the type  $SP(1, n)$  and  $F_4^{-20}$ . To obtain as complete a classification of linear representations from the work of Corlette and Gromov-Schoen as obtained by Margulis in [M2], one also needs cohomology vanishing theorems due to Matsushima-Murakami and Raghunathan [MM, Rg] as well as some results from [M2]. Later, Jost-Yau [JY] and Mok-Siu-Yeung [MSY] showed that one could reprove Margulis' theorems using harmonic map technology, at least for cocompact lattices.

Some attempts have been made to use harmonic map techniques to prove Zimmer's cocycle superrigidity theorem for this broader class of groups  $G$ , i.e. for  $G$  with no compact factors and property (T) of Kazhdan, see [CZ, KS3]. The results in those papers require additional assumptions on the cocycle which make them difficult to apply. Some striking applications, particularly to actions on low dimensional manifolds, are given in each paper.

In this paper, we prove a complete analogue of Zimmer's theorem for cocycles satisfying a certain weak integrability condition defined below. This allows us to generalize a huge class of results for actions of higher rank lattices to actions of lattices with property (T). Our contribution has two main aspects. First, we allow the cocycle to have target which is not already assumed semisimple. Second, we remove conditions concerning the image of the cocycle, mainly by removing any assumptions concerning fixed points at infinity. In both places our results are analogous to Zimmer's result, in higher rank, that the algebraic hull of the cocycle is reductive [Z8]. In our setting, the algebraic hull is not a well-defined invariant, so our proofs do not resemble the ergodic theoretic proofs very closely, though they are loosely inspired by the techniques of [M2, Z8].

The results in [CZ, KS3] can be thought of as extending those of [Co2, GS] to the infinite dimensional context. The papers [Co2, GS] apply to representations  $\rho : \Gamma \rightarrow GL(n, k)$  with the Zariski closure of  $\rho(\Gamma)$  reductive and the conditions needed in [CZ, KS3] are analogous to this. The arguments in [MM, Rg, M2] which reduce the case of general linear representations  $\rho$  to the case where the Zariski closure  $\rho(\Gamma)$  is reductive. In the finite dimensional setting, the key point, which follows from [MM, Rg], is vanishing of  $H^1(\Gamma, V)$  where  $V$  is a finite dimensional vector space which is  $\Gamma$  module. While the work we do here is analogous to the reduction in [MM, Rg, M2], in our setting the linear problem is replaced by a non-linear one. It is possible to reduce the problem we study to a question concerning  $H^1(\Gamma, F)$  where  $F$  is a topological vector space which is a non-trivial  $\Gamma$  module, but there is no inner product on  $F$  which behaves well under the  $\Gamma$  action. This forces us to use a non-linear method to show that  $H^1(\Gamma, F) = 0$ . In fact, the formulation in terms of vanishing of cohomology is not particularly useful in our context. See subsections 2.2 and 2.3 for more discussion.

In the remainder of the introduction, we formulate our main results and some applications. In subsection 1.1, we state the result on existence and rigidity of harmonic maps that we use to prove our cocycle superrigidity theorem. This is of independent interest. In subsection 1.2 we describe our cocycle superrigidity theorem and in subsection 1.3 we highlight some applications of these results. In the second half of this announcement, we sketch the proofs of our harmonic map results, first describing some reductions of the problem in subsection 2.2 and then outlining the main technical step in subsection 2.3.

For the remainder of this paper  $G$  is a semisimple Lie group with no compact factors and property (T) of Kazhdan and  $\Gamma < G$  is a lattice.

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**1.1. Harmonic map results.** We let  $\tilde{M} = G/K$  where  $K$  is a maximal compact subgroup be the symmetric space associated to  $G$ . Throughout this section  $M$  will be a compact locally symmetric space with universal cover  $\tilde{M}$  and  $\pi_1(M) = \Gamma$  will be a cocompact lattice. The results stated here probably also hold for  $M$  of finite volume, but there is a fundamental problem of existence of harmonic maps, or even finite energy maps, which remains unresolved. See [JL, JY, Sa] for some partial results in this direction.

We will be considering a harmonic map problem for certain representation  $\rho : \Gamma \rightarrow \text{Isom}(X)$  where  $X$  is an aspherical Hilbert manifold. We now describe the class of Hilbert manifolds we consider, their isometry groups and the condition we need on  $\rho(\Gamma)$ .

Let  $N$  be a simply connected, aspherical, homogeneous Riemannian manifold. (Here we reserve the term Riemannian manifold for the finite dimensional case and refer to the infinite dimensional analogues as Hilbert manifolds.) We can write  $N = H/C$  where  $H = \text{Isom}(N)$  and  $C$  is a maximal compact subgroup. We assume that  $C$  does not contain any subgroups which are normal in  $H$ , and thus the  $H$  action on  $H/C$  is effective. To simplify exposition, we will assume that  $H$  is the real points of a linear real algebraic group, then we can represent  $H$  as a subgroup of  $GL(m, \mathbb{R})$  for some large  $m$ . Let  $(S, \mu)$  be a probability measure space. For  $\phi_1, \phi_2 : S \rightarrow H/C$  measurable, we define

$$d_X(\phi_1, \phi_2)^2 = \int_S d_N^2(\phi_1(s), \phi_2(s)) d\mu(s)$$

Let  $\phi_0 : S \rightarrow G/K$  be  $\phi_0(S) = [C]$  and define  $X := L^2(S, \mu, H/C)$  to be the set of measurable maps  $\phi : S \rightarrow H/C$  with  $d(\phi, \phi_0) < \infty$ . If  $S$  is a finite set, then  $X$  is a finite product  $\prod_{s \in S} H/C$  where the metric on each factor is scaled by  $\mu(s)$ . In general, we call  $X$  a *continuum product* of copies of  $H/C$  and  $X$  is a simply connected, complete, homogeneous, aspherical, Hilbert manifold. If  $N = H/C$  has non-positive curvature

then so does  $X$ , and if  $N$  has mixed curvature so does  $X$ . We describe a subgroup of the isometry group of  $X$ . Let  $T : S \rightarrow S$  be a measure preserving transformation and  $f : S \rightarrow H$  a measurable map with

$$(1) \quad \int_S (\ln^+ \|f(s)\|)^2 < \infty$$

where here we are taking the matrix norm of  $f(s)$  defined by the embedding of  $H$  into  $GL(m, \mathbb{R})$ . We let  $L_{\ln}^2(S, \mu, H)$  be the set of maps which satisfy equation (1). Then we can define an isometry  $F$  of  $X$  by

$$F(\phi)(s) = f(T^{-1}s)\phi(T^{-1}s)$$

for  $\phi \in X$ . We denote the group of such isometries by  $\text{Isom}_\mu(X)$ . We remark that there is a special case of this construction where we take  $S$  to be a compact Riemannian manifold,  $\mu$  to be the Riemannian volume and  $L^2(S, \mu, SL(n, \mathbb{R})/SO(n))$  to be the measurable,  $L^2$  sections of the  $SL(n, \mathbb{R})/SO(n)$  bundle over  $S$  associated to the frame bundle, so  $n = \dim(S)$ . If  $\Gamma$  acts smoothly on  $S$  preserving  $\mu$  then the derivative of the action yields an isometric action of  $\Gamma$  on  $L^2(S, \mu, SL(n, \mathbb{R})/SO(n))$ . For this example,  $X$  is usually referred to as the “space of  $L^2$ -Riemannian metrics” on  $S$ , see for example [KS3]. We prove the following [FH1].

**Theorem 1.1.** *Let  $M, \tilde{M}, \Gamma, X$  as above. Then given  $\rho : \Gamma \rightarrow \text{Isom}_\mu(X)$ , there exists a  $\Gamma$  equivariant harmonic map  $f : M \rightarrow X$ . Furthermore  $f$  is either constant or totally geodesic.*

The assumption that  $\rho(\Gamma)$  lands in  $\text{Isom}_\mu(X)$  may at first seem similar to the usual assumption in the theory of equivariant harmonic maps that the action is *reductive*. However, we are not assuming reductivity in the standard sense. It is instead a consequence of Theorem 1.1 that the action must be reductive. In general,  $\text{Isom}_\mu(X)$  may be a proper subgroup of the full isometry group of  $X$ , but there are important cases where it is the whole isometry group. In particular, if  $H/C$  is an irreducible Riemannian symmetric space of the non-compact type, then we can show  $\text{Isom}(X) = \text{Isom}_\mu(X)$ . So, in this case, we are making no assumption at all.

The idea to prove cocycle superrigidity result by studying harmonic maps into infinite dimensional spaces first appears in work of Korevaar and Schoen [KS1, KS2, KS3]. The foundations of a theory of harmonic maps with infinite dimensional target spaces are laid out in those papers and also, independently and from a somewhat different viewpoint, in work of Jost [Jo1, Jo2, Jo3].

In [KS2, KS3], Korevaar and Schoen introduce the notion of a zero vanishing theorem. This is a method for proving the existence of equivariant harmonic maps into non-positively curved spaces of finite or infinite dimension without assuming reductivity of the action. Zero vanishing theorems are developed further in work of Gromov, Izeki-Nayatani and Schoen-Wang [Gr3, IN, SW]. The general philosophy of zero vanishing theorems, as well as all the results mentioned, work in a context where

one knows a priori that any equivariant harmonic map must be constant, and the method in all cases is to produce the resulting fixed point for the action without first invoking an existence result. Though our work is similar to these works in that we produce equivariant harmonic maps without any reductivity assumption, the method is necessarily quite different because we need to consider a setting where non-constant harmonic maps exist. A key step in our proofs, and our primary analytic innovation, is what one might call a “relative zero vanishing theorem”, see Theorem 2.5 and the discussion in subsection 2.3. This result is the key mechanism for allowing us to produce non-constant equivariant harmonic maps when the technique of zero vanishing theorems cannot apply.

There is another point of view, introduced in [Gr2] and used in [CZ] of studying foliated harmonic maps instead of harmonic maps with infinite dimensional targets. It is probably possible to formulate and prove our results from that point of view as well. For the problems that concern us, the difference in these points of view is merely whether one thinks of the variable in  $S$  as belonging to the domain or the range. That is, one may try to solve a family of harmonic map problems parameterized by  $S$  or try to solve a harmonic map problem into a product of spaces that is parameterized by  $S$ . Further work using this point of view to study superrigidity questions was pursued by Benoit Rivet in an unpublished Ph.D. thesis [Ri].

**1.2. Cocycle super-rigidity results.** In this section we formulate our main result concerning cocycle super-rigidity. In what follows, we let  $G$  be a semisimple Lie group with no compact factors and property (T) of Kazhdan,  $\Gamma < G$  a lattice and  $(S, \mu)$  a probability measure space.

We recall some basic notions concerning cocycles. Given a group  $D$ , a space  $S$  and an action  $\rho : D \times S \rightarrow S$ , we define a *cocycle over the action* as follows. Let  $L$  be a group, then a cocycle is a map  $\alpha : D \times S \rightarrow L$  such that  $\alpha(g_1 g_2, s) = \alpha(g_1, g_2 s) \alpha(g_2, s)$  for all  $g_1, g_2 \in D$  and all  $s \in S$ . The regularity of the cocycle is the regularity of the map  $\alpha$ . If the cocycle is measurable, we only insist on the equation holding almost everywhere in  $S$ . Note that the cocycle equation is exactly what is necessary to define a skew product action of  $D$  on  $S \times L$  or more generally an action of  $D$  on  $S \times Y$  by  $d \cdot (x, y) = (dx, \alpha(d, x)y)$  where  $Y$  is any space with an  $L$  action.

We say two cocycles  $\alpha$  and  $\beta$  are *cohomologous* if there is a map  $\phi : S \rightarrow L$  such that  $\alpha(d, s) = \phi(ds)^{-1} \beta(d, s) \phi(s)$ . Again we can define the cohomology relation in any category, depending on how much regularity we seek or can obtain on  $\phi$ . A cocycle is called *constant* if it does not depend on  $s$ , i.e.  $\alpha_\pi(d, s) = \pi(d)$  for all  $s \in S$  and  $d \in D$ . One can easily check from the cocycle equation that this forces the map  $\pi$  to be a homomorphism  $\pi : D \rightarrow L$ . When  $\alpha$  is cohomologous to a constant cocycle  $\alpha_\pi$  we will often say that  $\alpha$  is cohomologous to the homomorphism  $\pi$ . The cocycle superrigidity theorems imply that many cocycles are cohomologous to constant cocycles, at least in the measurable category.

Before stating our results, we require one more definition.

**Definition 1.2.** Let  $D$  be a locally compact group,  $(S, \mu)$  a standard probability measure space on which  $D$  acts preserving  $\mu$  and  $L$  a normed topological group. We call a cocycle  $\alpha : D \times S \rightarrow L$  over the  $D$  action  $L^2$  if for any compact subset  $M \subset D$ , the function  $Q_{M, \alpha}(x) = \sup_M \ln^+ \|\alpha(m, x)\|$  is in  $L^2(S)$ .

Here as above, we take the target of our cocycles,  $H$ , to be the real points of a real algebraic group in order to have convenient norms to work with. This is not necessary, but does simplify notations and discussion. We prove the following superrigidity theorems for cocycles [FH1].

**Theorem 1.3.** Let  $G, S, \mu, H$  be as above. Assume  $G$  acts ergodically on  $S$  preserving  $\mu$ . Let  $\alpha : G \times S \rightarrow H$  be an  $L^2$ , Borel cocycle. Then  $\alpha$  is cohomologous to a cocycle  $\beta$  where  $\beta(g, s) = \pi(g)c(g, s)$ . Here  $\pi : G \rightarrow H$  is a continuous homomorphism and  $c : G \times S \rightarrow C$  is a cocycle taking values in a compact group centralizing  $\pi(G)$ .

**Theorem 1.4.** Let  $G, \Gamma, S, H$  and  $\mu$  be as above. Assume  $\Gamma$  acts ergodically on  $S$  preserving  $\mu$ . Assume  $\alpha : \Gamma \times S \rightarrow H$  is an  $L^2$ , Borel cocycle. Then  $\alpha$  is cohomologous to a cocycle  $\beta$  where  $\beta(\gamma, s) = \pi(\gamma)c(\gamma, s)$ . Here  $\pi : G \rightarrow H$  is a continuous homomorphism of  $G$  and  $c : \Gamma \times S \rightarrow C$  is a cocycle taking values in a compact group centralizing  $\pi(G)$ .

The proofs of these results proceed as follows. For  $\Gamma$  cocompact, Theorem 1.4 is a direct consequence of Theorem 1.1. We can then deduce Theorem 1.3 from Theorem 1.4 using Iozzi's thesis and the method described in the paper of Corlette and Zimmer [CZ, I]. It is then standard to prove Theorem 1.4 from Theorem 1.3 by inducing actions and cocycles. The only non-trivial step in that process is proving that the  $L^2$  condition is preserved by induction, but this is easy to check in the cases that concern us.

We remark that the assumption of ergodicity in Theorems 1.3 and 1.4 are not exactly necessary, but that without it the statements become more complicated. See [FMW] for a method of deducing the non-ergodic case from the ergodic one, as well as for precise statements in the non-ergodic case. In our setting, we prove the non-ergodic statements directly from Theorem 1.1, though the proof of that theorem often requires that one analyze the actions on one ergodic component at a time.

**1.3. Some applications.** In this subsection, we discuss some applications of Theorems 1.3 and 1.4. The gist of this section is that most results concerning smooth actions of higher rank groups that were proven using super-rigidity for cocycles now apply more broadly. This includes all of the results in [Z1, Z2, Z3, Z4, Z5, Z6, Z7].

Some notable results, such as the long string of local rigidity results beginning with [Hu] and culminating in the work [FM1, FM2, FM3] are not quite established by our methods. One step in the proof of these results is not entirely in place: all methods used for showing smoothness along dynamical foliations depend on the presence of higher rank abelian subgroups, see particularly [Hu, KaSp]. While it

should be possible to replace these methods by the methods developed by the second author in [H], we are instead pursuing a project to prove local rigidity directly using the estimates used in the proofs of Theorems 1.1 and 1.8 and the criterion for local rigidity introduced in [F]. We will discuss this approach elsewhere. We note here that the  $C^{3,0}$  local rigidity of [FM3, Theorem 1.1] is an immediate consequence of the methods here.

More precisely let  $G$  be a (connected) semisimple Lie group with no compact factors and property (T) of Kazhdan, and  $\Gamma < G$  is a lattice. Then combining Theorems 1.3 and 1.4 with the methods of [FM1, FM2, FM3] yields:

**Theorem 1.5.** *Let  $\rho$  be a quasi-affine action of  $G$  or  $\Gamma$  on a compact manifold  $X$ . Then the action is  $C^{3,0}$  locally rigid.*

For the reader's benefit, we recall the definition of quasi-affine from [FM3]. We remark here that our definition of quasi-affine implies that all actions considered in Theorem 1.5 are volume preserving. First recall that, for any Lie group  $H$  and any closed subgroup  $\Lambda$ , an *affine diffeomorphism*  $d$  of  $H/\Lambda$  is one covered by a diffeomorphism  $\tilde{d}$  of  $H$  of the form  $\tilde{d} = A \circ T_h$  where  $A$  is an automorphism of  $H$  such that  $A(\Lambda) = \Lambda$  and  $T_h$  is left translation by  $h \in H$ .

**Definition 1.6. a)** *Let  $H$  be a connected real algebraic group,  $\Lambda < H$  a cocompact lattice. Assume a topological group  $G$  acts continuously on  $H/\Lambda$ . We say that the  $G$  action on  $H/\Lambda$  is affine if every element of  $G$  acts via an affine diffeomorphism.*

**b)** *More generally, let  $M$  be a compact manifold. Assume a group  $G$  acts affinely on  $H/\Lambda$ . Choose a Riemannian metric on  $M$  and a cocycle  $\iota : G \times H/\Lambda \rightarrow \text{Isom}(M)$ . We call the skew product action of  $G$  on  $H/\Lambda \times M$  defined by  $d \cdot (x, m) = (d \cdot x, \iota(d, x) \cdot m)$  a quasi-affine action.*

For  $\Gamma$  and  $G$  as above, all affine actions are classified by results in [FM1].

We note here one other application. The action on the space of metrics corresponds to studying the derivative cocycle, and this is the primary mechanism for many applications. Here we state results that use other cocycles over the action to motivate our more general results. The result we state generalizes a result due to the first author and Zimmer.

**Theorem 1.7** ([FZ]). *Let  $\Gamma < G$  be a lattice, where  $G$  is a noncompact simple Lie group with property (T) of Kazhdan. Suppose  $\Gamma$  acts analytically and ergodically on a compact manifold  $S$  preserving volume and an analytic connection. Then either:*

- (1) *the action is isometric and  $S = K/C$  where  $K$  is a compact Lie group,  $C$  is a closed subgroup and the action is by right translation via  $\rho : \Gamma \hookrightarrow K$ , a dense image homomorphism, or*
- (2) *there exists an infinite image linear representation  $\sigma : \pi_1(M) \rightarrow GL_n(\mathbb{R})$ , such that the algebraic automorphism group of the Zariski closure of  $\sigma(\pi_1(M))$  contains a group locally isomorphic to  $G$ .*

**Remarks:**

- (1) The analytic connection in the statement of the theorem can be replaced by any analytic rigid geometric structure in the sense of [Gr1]. For  $G$  actions a similar result is proved in [Gr1].
- (2) Finer information on the representation in conclusion (2) is obtained in [FZ]. To obtain the same information here, we need to extend our results to cover cocycles into groups defined over other local fields. This is work in progress.

This theorem is proven by studying cocycles associated to linear representations of the fundamental group of  $M$ . Other theorems proven by studying that type of cocycle are also easily generalized using our results modulo the remark in (2) above. See for example [FW, Sch] and references there.

**1.4. Vanishing Cohomology.** Our methods also allow us to deduce new cohomology vanishing results for certain representations for the groups  $\Gamma$  as defined above. Here  $\Gamma$  is a cocompact lattice in  $G$  as above. We prove the following theorem [FH2].

**Theorem 1.8.** *Let  $\sigma : G \rightarrow GL(V)$  be a finite dimensional representation of  $G$  and  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  a unitary representation of  $\Gamma$  on a separable Hilbert space  $\mathcal{H}$ . Then  $H^1(\Gamma, \sigma \otimes \pi) = 0$ .*

**Remarks:**

- (1) For certain representations of  $G$  which are of the type described, this result is known. Namely it is known for  $G$  representations which are induced from *finite dimensional*  $\Gamma$  representations of this kind. These representations are  $\Gamma$  representations which induce to *automorphic* representations of  $G$ . Proofs given in that context can be made to work somewhat more generally but do not appear to prove the result stated here, see [BW] and references there.
- (2) The proof of Theorem 1.1 uses an estimate that is also used in the proof of Theorem 1.8. Theorem 1.8 is proven by using a Bochner type estimate to give a lower bound on a Laplacian on the associated  $V \otimes \mathcal{H}$  bundle over  $M$ .
- (3) A key step in the proof of Theorem 1.1 is proving a generalization of Theorem 1.8 that requires a non-linear heat flow argument.

## 2. ON PROOFS

In this section, we discuss the proofs of Theorem 1.1 and Theorem 1.8. In the first subsection, we describe the relation between the results in subsection 1.1 and those in subsection 1.2, as well as stating some results which are used in the proof. The key point is that cocycles are simply a method for writing the action on  $L^2(S, \mu, H/C)$  in coordinates. In the second subsection, we sketch the reduction the proof of Theorem 1.1 to a special case, whose proof we outline in the final subsection.

For simplicity, throughout this section, we will assume that the  $\Gamma$  action on  $(S, \mu)$  is ergodic. This is tantamount to saying that there is no non-trivial  $\Gamma$  invariant splitting



of  $X$  as a product  $X = L^2(S_1, \mu|_{S_1}, H/C) \times L^2(S_2, \mu|_{S_2}, H/C)$  where  $S_1, S_2$  are subsets of  $S$  with  $S_1 \cup S_2 = S$  up to sets of measure zero.

**2.1. Cocycles as coordinates for actions on  $L^2(S, \mu, H/C)$ .** In this section, we set up the general method of translating between properties of  $H$  valued cocycles over the  $\Gamma$  action on  $S$  and properties of the  $\Gamma$  action on  $X = L^2(S, \mu, H/C)$  coming from a homomorphism  $\rho : D \rightarrow \text{Isom}_\mu(X)$ .

Given  $\phi \in L^2(S, \mu, H/C)$ , our assumption on  $\rho$  implies that each  $\rho(\gamma)$  can be written as

$$(\rho(\gamma)\phi)(s) = f_\gamma(\gamma^{-1}s)\phi(\gamma^{-1}s)$$

where  $f : S \rightarrow H$  is in  $L^2_{\text{in}}(S, \mu, H)$ . The fact that  $\rho$  defines a  $\Gamma$  action immediately implies that the map  $\alpha_\rho(\gamma, s) = f_\gamma(s)$  is a cocycle over the  $D$  action on  $S$ . That  $\alpha$  is an  $L^2$  cocycle is immediate. Conversely, given an  $L^2$  cocycle  $\alpha$ , we can define an action  $\rho$  using these equations. Similar statements can be made concerning action/cocycles of continuous groups, though some care needs to be taken concerning continuity.

The following three lemmas translate geometric properties of  $\rho$  into algebraic and dynamical properties of  $\alpha_\rho$ . The first is almost trivial and does not really depend on any assumption on the acting group  $D$  or on ergodicity of the  $D$  action on  $S$ .

**Lemma 2.1.** *Let  $\rho : D \rightarrow \text{Isom}_\mu(L^2(S, \mu, H/C))$ . Then  $\pi$  has a fixed point if and only if  $\alpha_\rho$  is  $L^2$  cohomologous to a cocycle taking values in (a conjugate of)  $C$ .*

If the action is not ergodic, the cocycle may take values in different conjugates of  $C$  over different ergodic components of the action on  $S$ .

For the remaining two lemmas, the assumption of ergodicity of the  $D$  action on  $X$  is important. Variants can be stated if the action is not ergodic, but their formulation is considerably more complicated.

For the following lemma, we need the assumption that  $H$  is a reductive group. Variants of this Lemma continue to hold if  $H/C$  is replaced by any proper CAT(0) space. A variant of this lemma was discovered independently by Furman and Monod in their work on cocycle super-rigidity for products of groups [FuMo].

**Lemma 2.2.** *Let  $\rho : \Gamma \rightarrow \text{Isom}_\mu(L^2(S, \mu, H/C))$ . There is a  $\rho$  invariant equivalence class of geodesic rays if and only if there is a parabolic subgroup  $P < H$  and there is a  $\rho$  invariant subspace of  $L^2(S, \mu, H/C)$  isometric to  $L^2(S, \mu, P/C' \cap P)$  where  $C'$  is a conjugate of  $C$ . This is equivalent to  $\alpha_\rho$  being  $L^2$  cohomologous to a cocycle taking values in  $P$ .*

The proof of the final lemma is essentially contained in the paper of Corlette and Zimmer [CZ]. For this lemma, we require our standing assumptions on  $\Gamma, G$  and  $M$ , but do not require that  $H$  is reductive.

**Lemma 2.3.** *Let  $\pi : \Gamma \rightarrow \text{Isom}_\mu(L^2(S, \mu, H/C))$ . There is a  $\rho$  equivariant totally geodesic, harmonic map from  $\tilde{M}$  to  $L^2(S, \mu, H/C)$  if and only  $\alpha_\rho$  is  $L^2$ -cohomologous*

to a cocycle  $\beta(\gamma, s) = \pi(\gamma)c(\gamma, s)$ . Here  $\pi : G \rightarrow H$  is a continuous homomorphism and  $c : \Gamma \times S \rightarrow C$  is a cocycle taking values in a compact group centralizing  $\pi(G)$ .

The proofs of all three lemmas use a standard argument to play off ergodicity of the  $\Gamma$  action on  $S$  against tameness of various actions of algebraic groups.

**2.2. First reduction for the harmonic map problem.** In this subsection we describe how to reduce the proof of Theorem 1.1 to a special harmonic map problem whose solution is described in the next subsection. As  $H$  is a Lie group, we know that  $H = L \ltimes U$  where  $U$  is nilpotent and  $L$  is reductive. Our assumptions on  $H/C$  guarantee that  $L$  has no compact simple factors and that (up to conjugation)  $C$  is a subgroup in  $L$ . In this context  $H/C$  is a fiber bundle with fiber  $U$  over  $L/C$  and this makes  $X$  into a fiber bundle

$$\begin{array}{ccc} L^2(S, \mu, H/C) & \xleftarrow{i} & F(S, \mu, U) \\ \pi \downarrow & & \\ L^2(S, \mu, L/C) & & \end{array}$$

where  $F(S, \mu, U) = \{f \in L^2(S, \mu, H/C) \mid f(y) \in U \text{ a.e.}\}$ . We note here that  $L^2(S, \mu, U) \subsetneq F(S, \mu, U)$  unless the  $L$  action on  $U$  is trivial. The method of proof is to show that the  $\Gamma$  action preserves a totally geodesic copy of  $L^2(S, \mu, L/C)$  in  $L^2(S, \mu, H/C)$ . To do this, we can use the fibered structure of  $U$  to inductively reduce to the case where  $U = \mathbb{R}^n$ . I.e. to studying the case where the above fibration is replaced by:

$$\begin{array}{ccc} L^2(S, \mu, H/C) & \xleftarrow{i} & F(S, \mu, \mathbb{R}^n) \\ \pi \downarrow & & \\ L^2(S, \mu, L/C) & & \end{array}$$

where  $F(S, \mu, \mathbb{R}^n) = \{f \in L^2(S, \mu, H/C) \mid f(y) \in \mathbb{R}^n \text{ a.e.}\}$ . We show that  $\rho$  projects to a homomorphism  $\bar{\rho} : \Gamma \rightarrow \text{Isom}(L^2(S, \mu, L/K))$ . We can study  $\rho$  first by using results of Korevaar and Schoen [KS3], namely the following:

**Theorem 2.4** (Korevaar-Schoen). *Let  $M, \tilde{M}, \Gamma$  and  $L$  as above and  $X = L^2(S, \mu, L/K)$ . Then either*

- a. *there exists a  $\Gamma$  equivariant harmonic map  $\phi : \tilde{M} \rightarrow L^2(S, \mu, L/K)$  which is either constant or totally geodesic, or*
- b. *there is an asymptotic equivalence class  $[c]$  of rays in  $L^2(S, \mu, L/K)$  invariant under  $\Gamma$ .*

**Remarks:**

- (1) Korevaar and Schoen only claim this theorem in the special case of actions on the space of metrics. It is easy to check that their proof goes through in the generality given here.

- (2) More is true, they construct a functional  $I$ , related to energy, which is eventually decreasing along any ray in the equivalence class  $[c]$ .
- (3) One can reformulate the existence of  $[c]$  in terms of the existence of a fixed point on  $\partial X$ .

Our proof of Theorem 1.1 in the special case where  $H$  is reductive amounts to eliminating case  $b$ . of Theorem 2.4. To do this, we apply Lemma 2.2, which produces a parabolic subgroup  $P < H$  such that  $\rho(\Gamma)$  preserves a subspace of the form  $L^2(S, \mu, P/(P \cap C'))$  where  $C'$  is a conjugate of  $C$ . We can, in fact, choose  $P$  minimal with this property. This uses the fact that stabilizers of rays in  $L/C$  are exactly (conjugacy classes of) parabolic subgroups which are ordered by inclusion. Now  $P$  is of the form  $L' \ltimes U'$  where  $L'$  is reductive and  $U'$  is unipotent. We can then repeat the discussion above, replacing  $H$  with  $P$ . However, when we return to apply Theorem 2.4 again, we are forced to be in case (a) by our assumption on minimality of  $P$ . Some care is required in this reduction as  $L'$  may have compact simple factors and additional arguments are required in that case.

Combining the reductions we have discussed so far, we can show that the problem of finding a harmonic map into  $L^2(S, \mu, H/C)$  reduces to the case where this space has the structure:

$$\begin{array}{ccc} L^2(S, \mu, H/C) & \xleftarrow{i} & F(S, \mu, \mathbb{R}^n) \\ \pi \downarrow & & \\ L^2(S, \mu, L/C) & & \end{array}$$

and we assume there is a  $\Gamma$  equivariant harmonic map  $f : \tilde{M} \rightarrow L^2(S, \mu, L/C)$ . Note that there is an embedding of  $L < H$  which defines an embedding  $L/C \subset H/C$  and an embedding  $L^2(S, \mu, L/C) \subset L^2(S, \mu, H/C)$  and that we can use this embedding to explicitly trivialize the bundle structure on  $L^2(S, \mu, H/C)$ . Showing that any harmonic map into this bundle takes values in a translate of  $L^2(S, \mu, L/C)$  is formally equivalent to vanishing of  $H^1(\Gamma, F(S, \mu, \mathbb{R}^n))$  where  $F(S, \mu, \mathbb{R}^n)$  becomes a  $\Gamma$  module by pulling back the vector bundle over  $\tilde{M}$  by  $f$  and then noting that it descends to a vector bundle on  $M$  by  $\Gamma$  equivariance of  $f$ . The  $\Gamma$  module structure on  $F(S, \mu, \mathbb{R}^n)$  can be given more explicitly, since under our assumptions, one can write the  $\Gamma$  action on  $L^2(S, \mu, H/C)$  as

$$\gamma \cdot (\phi_1(s), \phi_2(s)) = (\beta(\gamma, s)\phi_1(s), \beta(\gamma, s)\phi_2(s) + h(\gamma, s))$$

where  $\phi = (\phi_1, \phi_2)$  with  $\phi_1$  taking values in  $L$  and  $\phi_2$  takes values in  $\mathbb{R}^n$ . We know that  $\beta(\gamma, s) = \sigma(\gamma)c(\gamma, s)$  where  $\sigma$  is a  $G$  representation. Note that the representation  $\tau$  of  $\Gamma$  on  $L^2(S, \mu, \mathbb{R}^n)$  defined by  $(\gamma \cdot \phi)(s) = c(\gamma, \gamma^{-1}s)\phi(\gamma^{-1}s)$  is unitary. Here  $h$  is an  $F(S, \mu, \mathbb{R}^n)$  valued cocycle over the linear action of  $\Gamma$  determined by  $\beta$ .

If  $F(S, \mu, \mathbb{R}^n) = L^2(S, \mu, \mathbb{R}^n)$  or if  $h \in L^2(S, \mu, \mathbb{R}^n)$  then what is needed is exactly Theorem 1.8. However, it is only the case that  $F(S, \mu, \mathbb{R}^n) = L^2(S, \mu, \mathbb{R}^n)$  when the

map  $f$  is constant, in which case, vanishing of  $H^1(\Gamma, L^2(S, \mu, \mathbb{R}^n))$  is an immediate consequence of property (T) since the  $\Gamma$  representation on  $L^2(S, \mu, \mathbb{R}^n)$  is unitary in this case. When  $f$  is not constant, the fact that the distance we have on  $F(S, \mu, \mathbb{R}^n)$  is not induced by an inner product structure but by the embedding of  $F(S, \mu, \mathbb{R}^n)$  in  $L^2(S, \mu, \mathbb{R}^n)$  introduces profound difficulties. As a result we need to use a heat flow applied to the maps into  $L^2(S, \mu, H/C)$  which project to  $f$ . We describe this in the next section.

**2.3. Heat flow for maps into vector bundles.** In this section, we describe a heat flow method which we apply to complete the proof of Theorem 1.1. Here as above, we assume that  $\Gamma, G$  and  $M$  are as in Theorem 1.1. For the reader's convenience, we explicitly state the result we prove here:

**Theorem 2.5.** *Let  $H = L \ltimes \mathbb{R}^n$  be a Lie group where  $L$  is reductive and has no compact factors and  $C < L$  is a maximal compact subgroup. Let  $\rho : \Gamma \rightarrow \text{Isom}_\mu(L^2(S, \mu, H/C))$  be a homomorphism which projects to  $\bar{\rho} : \Gamma \rightarrow \text{Isom}_\mu(L^2(S, \mu, L/C))$  and let  $f : \tilde{M} \rightarrow L^2(S, \mu, L/C)$  be a  $\bar{\rho}$ -equivariant, totally geodesic, harmonic map. Then there is a lift of  $f$  to a map  $\tilde{f} : M \rightarrow L^2(S, \mu, H/C)$  such that  $\tilde{f}$  is harmonic,  $E(\tilde{f}) = E(f)$  and  $\tilde{f}$  takes values in some translate of  $L^2(S, \mu, L/C)$  in  $L^2(S, \mu, H/C)$ .*

**Remarks:**

- (1) We assume  $f$  is totally geodesic, since the case of  $f$  constant is easy by the final remarks of the last subsection.
- (2) It follows from Theorem 2.5 and a standard argument using Bochner estimates of Corlette [Co2], Jost-Yau [JY] or Mok-Siu-Yeung [MSY] that  $\tilde{f}$  is totally geodesic.
- (3) This theorem can be reformulated to say that  $H^1(\Gamma, F(S, \mu, \mathbb{R}^n)) = 0$ , but this point of view is not particularly helpful for understanding the proof.

We study this harmonic map problem by analyzing a heat flow that we construct here. The space on which the heat flow is constructed is

$$\mathcal{F} = \{g : G/K \rightarrow L^2(S, \mu, H/C) \mid g \text{ is } \rho\text{-equivariant and } \pi \circ g = f\}.$$

If we let  $\tau(g) = -\text{tr } \nabla dg$  be the tension field of  $g$  with respect to the natural metric on  $TM^* \otimes TY$ , then we can compute  $\tau(g) = (\tau_1(g), \tau_2(g))$  where our coordinates are from the splitting  $g^*TY = g^*TY' \oplus g^*TF$ . It is straightforward to check that the condition  $\pi \circ g = f$  implies that  $\tau_1(g) = 0$ . As noted above,  $X$  admits a global trivialization as  $Y = L^2(S, \mu, L/C) \times F(S, \mu, \mathbb{R}^n)$  in which we can write  $g(m) = (f(m), g_F(m))$ . Therefore the tension field is vertical with respect to our fibration. This is what allows us to construct a heat flow that preserves the class  $\mathcal{F}$  and which only alters  $g_F$ .

The key point for the rest of the argument is to understand the structure of  $TM^* \otimes g_F^*TF$ . First  $TF$  is a trivial  $L^2(S, \mu, \mathbb{R}^n)$  bundle over  $F$  and the associated bundle  $g_F^*L^2(S, \mu, \mathbb{R}^n)$  is easily seen to be a flat bundle where the holonomy is a

representation of the form  $\sigma \otimes \tau$  where  $\sigma$  is the restriction of a finite dimensional  $G$  representation and  $\tau$  is a unitary  $\Gamma$  representation. In fact, as described in the last subsection we can write the action of  $\Gamma$  on  $X$  in coordinates and then the  $\Gamma$  module structure on the bundle  $g_F^* L^2(S, \mu, \mathbb{R}^n)$  is given explicitly.

Short time existence of heat flow follows from an adaptation of standard arguments using implicit function theorems. Long time existence is then shown using the fact that  $g^* L^2(X, \mu, \mathbb{R}^n)$  is a flat bundle and the Eells-Sampson Bochner formula in a manner similar to that in [Co1]. Both of these steps are, of course, complicated by the fact that our target space is not finite dimensional.

To understand the long time behavior of the heat flow requires use of finer structure. In particular, we use another Bochner type estimate to show that the total energy of  $g_2$  goes to zero as  $t \rightarrow \infty$ . Once we know this, Theorem 2.5 is immediate, since this implies that  $g_2$  is constant.

The Bochner-type estimate we use in this context is the same one that is used to prove Theorem 1.8. In the context of Theorem 1.8, this estimate is used to produce a lower bound on the Laplacian on one forms of the form  $(\Delta u, u) > c\|u\|^2$  for some absolute constant  $c$ . This forces cohomology to vanish by [Mk, Proposition 1.3.1]. The same estimate applied in the non-linear setting implies  $\|\tau(g_2)\| > cE(g_2)$ . If  $g_2^t$  is the heat flow applied to  $g_2$  at time  $t$ , then this implies that

$$\frac{dE(g_2^t)}{dt} = -\|\tau(g_2^t)\|^2 < -cE(g_2^t)$$

which suffices to show that  $E(g_2^t)$  decreases exponentially quickly to zero.

The estimate we need follows by comparing two Laplacians on the  $L^2(X, \mu, \mathbb{R}^n)$  bundle on  $M$  or, more precisely, by taking the difference of Bochner formulae for these two Laplacians. Each Laplacian is associated to a connection. The first connection  $\nabla_1$  is simply the standard flat bundle connection with associated differential  $d_1$  and Laplacian  $\Delta_1$ . To discuss the second Laplacian we make a simplifying assumption. Namely, we assume that  $\mathbb{R}^n$  is an irreducible  $G$  module. In the full proof, algebraic arguments are used to reduce to this case. In this setting, we have a flat  $\mathbb{R}^n$  bundle over  $M$  with holonomy  $\sigma|_\Gamma$  and we can build an isomorphic  $\mathbb{R}^n$  bundle over  $M$  which is associated to the  $K$  bundle  $G/\Gamma$  over  $M$  as in [MM]. We show that this same process can be used to give a different structure on the  $L^2(S, \mu, \mathbb{R}^n)$  bundle. This other structure on the bundle gives rise to a second connection  $\nabla_2$  with associated differential  $d_2$  and Laplacian  $\Delta_2$ . Note that each Laplacian depends on a choice of metric on  $L^2(S, \mu, \mathbb{R}^n)$  which depends on the choice of metric on  $\mathbb{R}^n$ . To have the required estimate, we need the metric on  $\mathbb{R}^n$  to be *adapted* in the sense of Matsushima and Murakami [MM]. If  $G_{\mathbb{C}}$  is the complexification of  $G$  and  $\sigma_{\mathbb{C}}$  is the resulting representation on  $\mathbb{C}^n$ , then a metric on  $\mathbb{R}^n$  is adapted if it is the restriction of a Hermitian metric on  $\mathbb{C}^n$  which is invariant under the compact form  $G_c$  of  $G$  in  $G_{\mathbb{C}}$ . The metric on  $\mathbb{R}^n$  induced by an invariant metric on  $H/C$  is easily checked to have this property and it is also easy to check that this property is preserved by pulling

back by  $g_F$ . If one takes the general Eells-Sampson Bochner formulae for  $\Delta_1$  and  $\Delta_2$  on 1-forms then a computation similar to the one in [MM] implies that the difference  $\Delta_1 - \Delta_2$  is bounded below by a certain algebraic operator. This operator is shown to be strictly positive by Raghunathan in [Rg], which then forces  $\Delta_1$  to be strictly positive and completes our proofs. The interpretation of the Matsushima-Murakami Bochner formula as a difference of Eells-Sampson type Bochner formulas is implicit in [MM], but does not appear to have been noted explicitly before now.

By taking more care in the reductions than we have done here, it is possible to see that our argument shows that the functional  $I$  on the space  $X$  as defined by Korevaar and Schoen attains a minimum in  $X$ . Using this we can show that the harmonic map obtained here is a limit of a heat flow.

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